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A NOTE ON THE NAVARRO CONJECTURE FOR ALTERNATING GROUPS WITH ABELIAN DEFECT

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ABSTRACT. G.Navarro proposed (in [8]) a refinement of the unsolved McKay conjecture involving certain Galois automorphisms. The author verified this new conjecture for the alternating groups $A(\Pi)$ when $p = 2$ (see [7]). For odd primes p the conjecture is more difficult to study due the complexities in the p-local character theory. We consider the principal blocks of $A(\Pi)$ with an abelian defect group when p is odd: in this case the Navarro conjecture holds for p-singular characters.

1. McKay and Navarro conjectures

Let G be a finite group, $|G| = n$, p be a prime dividing n, D a Sylow p-group of G, and $N_G(D)$ the normalizer of D in G. Let $Irr(G)$ denote the irreducible characters of G, and $Irr_{p'}(G)$ the subset of characters whose degree is relatively prime to p. The following is a well-known conjecture.

Conjecture 1.1. (McKay, [1])

$$|Irr_{p'}(G)| = |Irr_{p'}(N_G(D))|.$$

Recently G. Navarro strengthened the McKay conjecture in the following way. All irreducible complex characters of G are afforded by a representation with values in the nth cyclotomic field $\mathbb{Q}_n/\mathbb{Q}$ (Lemma 2.15, [4]). Then the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ permutes the elements of $Irr(G)$. We denote the action of σ on $\chi \in Irr(G)$ by χ^σ. Then $\chi \in Irr(G)$ is *σ-fixed* if its values are fixed by σ, that is, $\chi^\sigma = \chi$. Let e be a nonnegative integer and consider $\sigma_e \in \mathcal{G}$ where $\sigma_e(\xi) = \xi^{p^e}$ for all p'-roots of unity ξ. Define $\mathcal{N}$ to be the subset of $\mathcal{G}$ consisting of all such σ_e. Let $Irr^\sigma_{p'}(G)$ and $Irr^\sigma_{p'}(N_G(D))$ be the subsets of $Irr_{p'}(G)$ and $Irr_{p'}(N_G(D))$ respectively fixed by $\sigma \in \mathcal{N}$.

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Conjecture 1.2. (Navarro, [8]) Let $\sigma \in \mathcal{N}$. Then

$$|Irr^{\sigma}_{p'}(G)| = |Irr^{\sigma}_{p'}(N_G(D))|.$$

The Navarro conjecture follows from the existence of a bijection ϕ from $Irr_{p'}(G)$ to $Irr_{p'}(N_G(D))$ that commutes with $\mathcal{N}$. That is, $\phi(\chi^{\sigma}) = \phi(\chi)^{\sigma}$ for all $\sigma \in \mathcal{N}$ and $\chi \in Irr_{p'}(G)$. The author verified in [6] that the Navarro conjecture holds for the alternating groups $A(\Pi)$ when $p = 2$. The verification when p is odd is more complicated since little is known about values of $Irr_{p'}(N_{A(\Pi)}(D))$. However in the special case that $A(\Pi)$ has an abelian defect group (equivalently $|\Pi| = n_0 + wp$ with $w < p$) this paper verifies that the Navarro conjecture holds for the p-singular characters of the principal block. The proof relies on results of P. Fong and M. Harris (see §4, [3]) on the irrationalities of the p-singular characters of $N_{A(\Pi)}(D)$.

2. A local-global bijection

2.1. p'-splitting characters of G.

Let $n \in \mathbb{N}$. A *partition* λ of n is a non-increasing integer sequence $(a_1, \cdots, a_m)$ satisfying $a_i \geq \cdots \geq a_m$ and $\sum_i a_i = n$. Then the *Young diagram* of λ is n nodes placed in rows such that the ith row of λ consists of a_i nodes. The (i, j)-*node* of λ lies in the ith row and jth column of the Young diagram. The (i, j)-*hook* h^{λ}_{ij} of $[\lambda]$ and consists of the (i, j)-node (or *corner* of h^{λ}_{ij}), all nodes in the same row and to the right of the corner, and all nodes in the same column and below the corner. The column-lengths of $[\lambda]$ form the *conjugate* partition λ^* of n. Partitions where $\lambda = \lambda^*$ are *self-conjugate*. Let $\lambda = \lambda^*$ and $\delta(\lambda) = \{\delta_{jj}\}$ be the set of *diagonal hooks* of λ i.e. $\delta_{jj} = h_{jj}$, which are necessarily odd. When there is no ambiguity we write $h^{\lambda}_{ij} = h_{ij}$.

Every λ is expressed uniquely in terms of its *p-core* λ^0 and *p-quotient* $(\lambda_0, \lambda_2, \cdots, \lambda_{p-1})$. The *p-core* λ^0 is the unique partition that results when all possible hooks of length p are removed from λ. The p-quotient $\langle\lambda\rangle$ is a p-tuple of (sub-)partitions which encode the p-hooks of λ.

Henceforth, let Π be a set of size n and $G = S(\Pi)$ and $G^+ = A(\Pi)$ be respectively the symmetric and alternating groups on Π. The elements of $Irr(G)$ are labeled by partitions $\{\lambda \vdash n\}$. Then $Irr(G^+)$ is obtained from $Irr(G)$ by restriction. If α is an irreducible character for some finite group J, and K is a subgroup of J, the notation $\alpha|_K$ indicates restriction of the subgroup K.

Theorem 2.1. *The irreducible characters of G^+ arise from those of G in two ways. If $\lambda \neq \lambda^*$ then $\chi_{\lambda}|_{G^+} = \chi_{\lambda^*}|_{G^+}$ is in $Irr(G^+)$. If $\lambda = \lambda^*$ then $\chi_{\lambda}|_{G^+}$ splits into two conjugate characters χ^+_{λ} and χ^-_{λ} in $Irr(G^+)$.*

The conjugacy classes κ of $S(\Pi)$ are labeled by cycle-types of permutations of n. If $\lambda = \lambda^*$ we let $\kappa_{\delta(\lambda)}$ be the conjugacy class determined by the cycle-type of $(\delta_{11}, \cdots, \delta_{dd})$. Then $\kappa_{\delta(\lambda)}$ splits into $\kappa_{\delta(\lambda),+}$ and $\kappa_{\delta(\lambda),-}$ when viewed as a class of G^+. Let $Irr^*(G)$ be the set of *splitting characters*, i.e. those that split into two conjugate characters when restricted to G^+. The following is a classical result of Frobenius (see e.g. Theorem (4A), [3]).

Theorem 2.2. *Suppose χ_λ is an irreducible character of G which splits on G^+. Let $g \in G^+$. Then $(\chi_{\lambda,+} - \chi_{\lambda,-})(g) \neq 0$ if and only if g is in $\kappa_{\delta(\lambda)}$. Moreover, $\chi_{\lambda,\pm}$ and $\kappa_{\delta(\lambda),\pm}$ may be labeled so that*

$$\begin{aligned}
\chi_\lambda^{\pm}(g) &= \tfrac{1}{2}[\epsilon_\lambda + \sqrt{\epsilon_\lambda \textstyle\prod_j \delta_{jj}}] \quad \text{if } g \in \kappa_{\delta(\lambda),\pm} \\
\chi_\lambda^{\pm}(g) &= \tfrac{1}{2}[\epsilon_\lambda - \sqrt{\epsilon_\lambda \textstyle\prod_j \delta_{jj}}] \quad \text{if } g \in \kappa_{\delta(\lambda),\mp}
\end{aligned}$$

where $\epsilon_\lambda = (-1)^{\frac{n-d}{2}}$.

By extension, $Irr^*(G^+)$ is the set of (pairs) of characters that arise from restricting elements of $Irr^*(G)$. Suppose $n = wp$, where $w < p$. By a condition of Macdonald (see [5]), the elements of $Irr_{p'}(G)$ are labeled by partitions for whom $\sum |\lambda_\gamma| = \omega$. Then the p'-splitting characters are labeled by self-conjugate partitions that satisify the Macdonald condition.

2.2. p'-splitting characters of H.

Let B be a p-block of G the defect group D and b the p-block of $N_G(D)$ which is the Brauer correspondent of B. Let ν be the exponential valuation of $\mathbb{Z}$ associated with p normalized so $\nu(p) = 1$. The height of the χ in B is the nonnegative integer $h(\chi)$ such that $\nu(\chi(1)) = \nu(|G|) - \nu(|D|) + h(\chi)$. The height of ξ in b is the nonnegative integer $h(\xi)$ such that $\nu(\xi) = \nu(|N_G(D)|) - \nu(|D|) + h(\xi)$. Let $M(B)$ and $M(b)$ be the characters of B and b of height zero. By the Nakayama conjecture a p-block B of G is parametrized by a p-core λ^0 so $\chi_\mu \in B$ if and only if $\lambda^0 = \mu^0$. In particular, $n = n_0 + wp$ where $n_0 = |\lambda_0|$. We suppose that B has abelian defect group D or equivalently $w < p$. Thus $\Pi = \Pi_0 \cup \Pi_1$ is the disjoint union of sets Π_0 and Π_1 of cardinality n_0 and wp. We may suppose $\Pi_1 = \Gamma \times \Omega$ where $\Gamma = \{1, 2, \cdots, p\}$ and Ω is a set of w elements. Let $X = S(\Gamma)$ and $Y = N_X(P)$ where P is a fixed Sylow p-subgroup of X. Note that the when B is a Sylow subgroup the p'-irreducible characters agree with the height zero characters.

We take D as the Sylow p-subgroup P^Ω of $S(\Pi_1)$ and set $H = N_G(D)$ so that $H = H_0 \times H_1$ with $H_0 = S(\Pi_0)$ and $H_1 = Y \wr S(\Omega)$. The Brauer

correspondent b of B in H has the form $b_0 \times b_1$ where b_0 is the block of defect 0 of H_0 parametrized by λ^0 and b_1 is the principal block of H_1.

Let λ be a partition of n with p-core λ^0 and p-quotient $\langle\lambda\rangle = (\lambda_0, \cdots, \lambda_{p-1})$ normalized as follows: if $\mu = \lambda^*$ then $\lambda_i = (\mu_{p-i-1})^*$. Let $p^* = \frac{p-1}{2}$. Then $\lambda = \lambda^*$ implies $\lambda_{p^*} = \lambda^*_{p^*}$. Let $Y^\vee = \{\xi_\gamma : 0 \leq \gamma \leq p-1\}$. The characters in $H^\vee$ have the form $\chi_\tau \times \psi_\Lambda$ where τ is a p-core partition and $\chi_\tau \in Irr(H_0)$ and $\psi_\Lambda \in Irr(H_1)$ and Λ is a mapping

$$Y^\vee \longrightarrow \{Partitions\}, \quad \xi_\gamma \mapsto \mu_\gamma,$$

such that $\sum_\gamma |\mu_\gamma| = w$. We also represent Λ by the p-tuple $(\mu_1, \cdots, \mu_p)$. Then $M(B)$ and $M(b)$ are in bijection via $f : \chi_\lambda \mapsto \chi_{\lambda^0} \times \psi_{\langle\lambda\rangle}$ (see [2] for details). Hence $Irr_{p'}(G)$ and $Irr_{p'}(H)$ are in bijection via $f = \cup_B f_B$. There is an induced bijection f^+ between $Irr_{p'}(G^+)$ and $Irr_{p'}(N_{G^+}(D))$. Let $sgn_H = sgn_G|_H$ and $sgn_Y = sgn_X|_Y$. If (f, σ) is an element of $H = Y \wr S(\Omega)$ with $f \in S(\Omega)$ and $f \in Y^\Omega$ and $\sigma \in S(\Omega)$, then

$$sgn_H(f, \sigma) = sgn_{S(\Omega)}(\sigma) \prod_{i \in \Omega} sgn_Y(f(i)).$$

Let $H^+ = N_{G^+}(D)$. Then Λ is a *splitting mapping* of H if ψ_Λ *splitting character* of H i.e. $(\psi_\Lambda)|_{H^+} = \psi_{\Lambda,+} - \psi_{\Lambda,-}$ where $\psi_{\Lambda,\pm} \in (H^+)^\vee$. Let * be the duality $\Lambda \mapsto \Lambda^*$ where $\Lambda^* : \xi_\gamma \mapsto (\lambda_{p-1-\gamma})^*$. The following is Proposition (4D) in [3].

Proposition 2.3. Let $\psi_\Lambda \in Irr(H)$. Then $sgn_H \psi_\Lambda = \psi_{\Lambda^*}$. In particular, ψ_Λ is a splitting character if and only if $\Lambda = \Lambda^*$.

Proposition 2.3 implies that map f^+ induced by f remains a bijection on splitting characters (and p'-splitting characters). That is, $Irr^*_{p'}(G^+)$ is in bijection with $Irr^*_{p'}(H^+)$. In particular, if $\lambda \neq \lambda^*$ then $\chi_\lambda|_{G^+} = \chi^*_\lambda|_{G^+}$ is mapped to $\psi_\Lambda|_{H^+} = \psi_{\Lambda^*}|_{H^+}$ and if $\lambda = \lambda^*$ then $\chi^\pm_\lambda$ maps to $\psi^\pm_\Lambda$.

3. Values of p-singular characters

We say λ is *p-singular* if $\lambda_{p^*} \neq \emptyset$ and $\lambda_i = \emptyset$ for all $i \in \{0, \cdots, p-1\} - p^*$. Then $\chi_\lambda \in Irr_{p'}(G)$ is p-singular if λ is. The notation $Irr_{p',sing}(G)$ denotes the p-singular p'-characters and $Irr_{p',sing}(G^+)$ is the restrictions to G^+. Then $Irr_{p',sing}(H)$ and $Irr_{p',sing}(H^+)$ are defined analogously. It is immediate from the definition of f^+ that $Irr_{p',sing}(G^+)$ and $Irr^*_{p',sing}(H^+)$ are in bijection. We show that f^+ commutes with the action of $\sigma \in \mathcal{N}$ on p-singular p'-characters by describing explicitly the relevant irrational character values.

In [6], the author describes how to obtain the set of diagonal hooks $\delta(\lambda)$ of a symmetric partition $\lambda = \lambda^*$ given just the p-core λ^0 and the p-quotient $\langle\lambda\rangle$. The following special case (Theorem 4.3 in [6]) is relevant to the goals of this paper.

Theorem 3.1. *Suppose λ^0 is empty and $(\emptyset, \cdots, \lambda_{p^*}, \cdots, \emptyset)$ such that $\lambda_{p^*} = (\lambda_{p^*})^*$ and $\delta(\lambda_{p^*})$=$(\delta'_{11}, \cdots, \delta'_{dd})$. Then $\delta(\lambda)$=$(\delta'_{11}p, \cdots, \delta'_{dd}p)$.*

A conjugacy class C of H is a *splitting class* if $C \subseteq H^+$ and $C = C_- \cup C_+$ is the union of two conjugacy classes of H^+. There is a bijection between splitting mappings Λ and splitting classes C_Λ of H (see pg.3491, [3]). The following is Proposition (4F) in [3].

Theorem 3.2. *Let $|\Pi| = wp$. Suppose Λ is a splitting mapping of $N_{S(\Pi)}(D)$ that equals its p-singular part i.e. $\Lambda = (\emptyset, \cdots, \lambda_{p^*}, \cdots, \emptyset)$. Let $(f, \sigma) \in N_{A(\Pi)}(D)^+$. Then $(\psi_{\Lambda,+} - \psi_{\Lambda,-})(f, \sigma) \neq 0$ if and only if $(f, \sigma) \in C_\Lambda$. Moreover, $\psi_{\Lambda,\pm}$ and $C_{\Lambda,\pm}$ may be labeled so that*

$$(\psi_{\Lambda,+} - \psi_{\Lambda,-})(f, \sigma) = \pm(\sqrt{\epsilon_p p})^d \sqrt{\epsilon_{\lambda_{p^*}} \prod_j \eta_{jj}}$$

for $(f, \sigma) \in C_{\Lambda,\pm}$, where, $\epsilon_{\lambda_{p^*}} = (-1)^{\frac{p-1}{2}}$, d is the number of diagonal nodes in λ_{p^*} and $\delta(\lambda_{p^*})$= $(\eta_{11}, \cdots, \eta_{dd})$.

Suppose $\sigma \in Gal(\mathbb{Q}_{|G^+|}/\mathbb{Q})$ is such that $\sigma(\xi) = \xi^{p^e}$ for some $e \in \mathbb{Z}^+$ and ξ is a p'-root of unity. We define $Irr_{p'}(B_1)$ and $Irr_{p'}(b_1)$ to be the p'-characters of the principal block B_1 of $A(\Pi)$ and its Brauer correspondent b_1 and $Irr_{p',sing}(B_1)$ and $Irr_{p',sing}(b_1)$ are defined by extension.

Theorem 3.3. *Let $A(\Pi)$ be the alternating group on Π and p is an odd prime such that $A(\Pi)$ has an abelian defect group. Let $\sigma \in \mathcal{N}$. Let B_1 be the principal block of $A(\Pi)$, $\chi \in Irr_{p'}(B_1)$ and b_1 its Brauer correspondent. Then the restriction of f^+ is a bijection between $Irr_{p',sing}(B_1)$ and $Irr_{p',sing}(b_1)$ that commutes with σ. That is, $f^+(\chi)^\sigma = f^+(\chi^\sigma)$.*

Proof. Since $A(\Pi)$ has abelian defect, and we are considering only the principal block, we can assume $|\Pi| = wp$. By the discussion above, we consider two cases.

(1) Suppose $\lambda \neq \lambda^*$. Then the restrictions $\chi_\lambda|_{G^+} = \chi_{\lambda^*}|_{G^+}$ are in bijection with $\psi_{\Lambda^*}|_{H^+} = \psi_{\Lambda^*}|_{H^+}$. Since the values of χ_λ are all rational, $\chi_\lambda|_*$ is σ-fixed. Since $N_{G^+}(X) = Y \wr S(\Omega)$ where $|\Omega| = p$, $\psi_\Lambda|_{H^+}$ is also σ-fixed.
(2) Suppose $\lambda = \lambda^*$. Upon restriction, the pair $\chi^\pm$ is in bijection with the pair $\psi^\pm_\Lambda$ via $\bar{f}$. It remains to show that the values of $\chi^\pm_\lambda$ and $\psi^\pm_\Lambda$ on the splitting classes $\kappa^\pm_{\delta(\lambda)}$ and $C^\pm_\Lambda$ are both exchanged

or fixed by σ. By Theorem 3.1, Theorem 2.2, and Theorem 3.2, $\sqrt{\eta_j p^d} = \sqrt{\delta_j}$. Since p is odd, $(wp - d) \equiv (p - 1 + w - d) \pmod 2$, so $\epsilon_{\lambda_{p^*}} \cdot \epsilon_p = \epsilon_\lambda$. This completes the proof.

□

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